free vibrations of a simply supported square plate consisting of two transversely isotropic layers was studied for several in-plane Young's moduli, in-plane-to-transverse shear moduli, and length-to-thickness ratios. The natural frequency results for the fundamental mode ($m=n=1$) are shown in Figs. 1 and 2.

As demonstrated by the figures, the in-plane-to-transverse shear moduli ratio has a very pronounced effect on the flexural frequencies, especially for thick composite plates. Even for plate length-to-thickness ratios of 20 or more, significant differences exist between the two predictions. It should be noted that, as the ratio of the in-plane Young's moduli increases, and therefore the ratio of the shear moduli increases since ($\overset{(1)}{A}=\overset{(2)}{A}$), the discrepancies become somewhat larger. Although not shown here for the sake of brevity, the discrepancies become much larger when higher modes are considered.

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Inviscid Flow Past a Sharp-Nosed Body with a Closed Finite Wake

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THE most commonly used model for the description of infinite Reynolds number flows past bodies whose shapes cause flow separation and the formation of a wake is the Helmholtz[1]-Kirchhoff[2] free streamline model. Batchelor[3] raises objections to both the open and the closed wake forms of this theory and proposes instead a closed wake model with a recirculating flow inside the wake which is such as to satisfy the requirement of continuity of pressure across the wake boundary. However, because the pressure is not constant along the boundary, much of the classical potential flow theory for wakes and cavities is not applicable. It is, consequently, much more difficult to determine the shape of the wake bubble than it is with the free streamline theory.

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In the present work a method is presented for computing, using Batchelor's model, the infinite Reynolds number flow for the case where the body is a two-dimensional shell forming the front part of a slender wake bubble.

Mainstream Irrotational Flow

Closed wake bubbles behind two-dimensional objects must necessarily terminate with a cusp to allow the velocity just outside the bubble boundary to be uniform. For simplicity, the present investigation is restricted to two-dimensional flows with fore and aft symmetry. In this case the wake bubble must have a cusp at its upstream end as well as its downstream end. The flow in the interior of the bubble is characterized by uniform vorticity of strength $-\omega$ in the upper half of the bubble and $+\omega$ in the lower half.[3,4] Thus, with distances and velocities made dimensionless with respect to the bubble length and the uniform stream velocity, respectively, the flow configuration is as shown in Fig. 1 (the ζ-plane), where t is the thickness ratio for the bubble, (x, y) are Cartesian coordinates and u_x and u_y are the velocity components in the x and y directions, respectively.

A major simplifying assumption is now made by supposing the bubble to be a slender one. That is, t is assumed small. With this simplification the magnitude of the velocity along the outside of the bubble boundary (ABC in Fig. 1) will be little different from the uniform stream value of unity. The flow in the ζ-plane may be determined through a mapping of the flow past a unit-diameter circular cylinder in the z-plane. The problem of finding the appropriate conformal transformation is, in general, a difficult one. However, since t is small, the required mapping may be regarded as a small perturbation of the mapping of the cylinder flow in the z-plane to the flow past a flat plate (AOC in Fig. 1) in the ζ-plane. A suitable form for the transformation is therefore

$$\zeta = z + \frac{1}{4z} - t\sum_{j=1}^{N} A_j (2z)^{1-2j} \tag{1}$$

where

$$\zeta = x + iy \quad z = re^{i\theta}$$

and where the $A_1 - A_N$ are real coefficients to be determined later by matching the irrotational flow to the inviscid flow in the wake.

The cylinder surface $r=½$ maps onto the bubble surface. Hence, equating real and imaginary parts of the transformation [Eq. (1)], putting $r=½$, and eliminating θ give the following equation for the shape of the bubble surface

$$y = t(1-x^2)^{½} \sum_{j=1}^{N} A_j U_{2j-2}(x) + 0(t^2) \tag{2}$$

where $U_{2j-2}(x)$ is the Chebyshev polynomial of the second kind in x of degree $2j-2$. Since, when $x=0$, $U_{2j-2}=(-1)^{j-1}$, and $y=t$, there is the requirement that

$$1 = \sum_{j=1}^{N} (-1)^{j-1} A_j \tag{3}$$

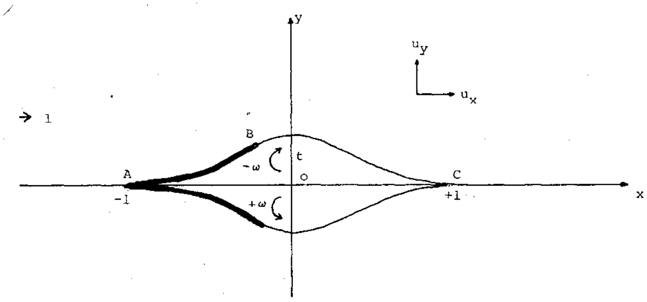

Fig. 1 Flow configuration (the ζ-plane).

The complex potential w for the flow in the z-plane is given by

$$w=z+(1/4z) \tag{4}$$

The complex velocity in the ζ-plane is then given by

$$\frac{dw}{d\zeta}=u_x-iu_y=\frac{dw}{dz} \Big/ \frac{d\zeta}{dz}$$

$$=\left(1-\frac{1}{4z^2}\right)\left[1-\frac{1}{4z^2}+2t\sum_{j=1}^{N}(2j-1)A_j(2z)^{-2j}\right]^{-1} \tag{5}$$

Now, at the points $\zeta=\pm 1$ [corresponding to $z=\pm ½$, to $0(t)$] the x component of the fluid velocity u_x will be different from zero (a consequence of the bubble having a cusp at each end). There is, therefore, the condition that

$$\sum_{j=1}^{N}(2j-1)A_j=0 \tag{6}$$

Equation (5) is then, to first order in t,

$$\frac{dw}{d\zeta}=1-2t\sum_{j=1}^{N}(2z)^{-2j}\sum_{k=1}^{j}(2k-1)A_k \tag{7}$$

For the magnitude of the velocity on the bubble boundary, Eq. (7) then gives to first order in t

$$q_p^2=\frac{dw}{d\zeta}\frac{d\bar{w}}{d\bar{\zeta}}=1-4t\sum_{j=1}^{N-1}T_{2j}(x)\sum_{k=1}^{j}(2k-1)A_k \tag{8}$$

where $T_{2j}(x)$ is the Chebyshev polynomial of the first kind in x of degree $2j$.

To summarize, Eq. (8) gives the magnitude of the velocity on the boundary of a bubble whose shape is described by Eq. (2).

Constant Vorticity Flow

For two-dimensional flow the magnitude of the vorticity is equal to minus the Laplacian of the stream function for the flow, viz., $-\nabla^2\psi$ [see Batchelor[5]]. From symmetry it is only necessary to consider the upper half of the wake bubble and, as already mentioned, the vorticity in this region is uniform and of strength $-\omega$. The stream function ψ for this part of the flow is, therefore, determined through the equation

$$\nabla^2\psi=(\partial^2\psi/\partial x^2)+(\partial^2\psi/\partial y^2)=\omega \tag{9}$$

together with the condition that ψ should be zero on the bubble boundary (ABC in Fig. 1) and on the axis of sysmmetry $y=0$.

A particular solution of Eq. (9) (although not satisfying the required boundary conditions) is $½\omega y^2$. If a term is added to this to render ψ zero on the boundary of the bubble, the following expression is obtained

$$\psi=½\omega y\left[y-t(1-x^2)^{½}\sum_{j=1}^{N}A_jU_{2j-2}(x)\right] \tag{10}$$

If Eq. (10) is substituted into Eq. (9) the remainder is found to be

$$½\omega yt(1-x^2)^{-3/2}\sum_{j=1}^{N}(2j-1)A_j\Big[2jxT_{2j-1}(x)$$
$$-(2j-1)T_{2j}(x)\Big]$$

It is not difficult to show that the maximum magnitude of this term occurs when $x=0$ and $y=t$ and is

$$½\omega t^2\sum_{j=1}^{N}(-1)^{j-1}(2j-1)^2A_j \tag{11}$$

It will be shown later that the sum in this relation remains finite (and not too large) whatever the value of N, so that the remainder represents an error of $0(t^2)$. It follows that equation (10) is, to first order in t, the required solution of Eq. (9).

The magnitude q_w of the velocity in the wake bubble is given by

$$q_w^2=(\partial\psi/\partial x)^2+(\partial\psi/\partial y)^2$$

and on the bubble boundary this gives

$$q_w^2=¼\omega^2t^2(1-x^2)\left[\sum_{j=1}^{N}A_jU_{2j-2}(x)\right]^2 \tag{12}$$

For q_w^2 to be the same order of magnitude in t as q_p^2it is clear that the vorticity must be of order $t^{-½}$. Thus putting

$$\omega=2\alpha t^{-½} \tag{13}$$

equation (12) becomes, after some rearrangement

$$q_w^2=½\alpha^2t\sum_{j=1}^{N}A_j\Big[2\sum_{k=1}^{j}A_kT_{2j-2k}(x)$$
$$-A_j-\sum_{k=1}^{N}A_kT_{2j+2k-2}(x)\Big] \tag{14}$$

Matching the Two Flows

As explained by Batchelor[3] the requirements of continuity of pressure across the dividing streamline lead to the following relation connecting the velocities just outside and inside the bubble boundary

$$q_p^2-q_w^2=\text{constant, e.g., } h \tag{15}$$

Substituting for q_p^2 and q_w^2 and equating terms involving x gives

$$\sum_{j=1}^{N-1}T_{2k}(x)\sum_{k=1}^{j}(2k-1)B_k$$
$$=½\sum_{j=1}^{N}\sum_{k=1}^{N}B_jB_kT_{2j+2k-2}(x)$$
$$-\sum_{j=2}^{N}\sum_{k=1}^{j-1}B_jB_kT_{2j-2k}(x) \tag{16}$$

where, for convenience, the following substitution has been made

$$A_j=4B_j/\alpha^2 \tag{17}$$

Equating terms not involving x gives for the constant h

$$h=1-\frac{8t}{\alpha^2}\sum_{j=1}^{N}B_j^2+0(t^2) \tag{18}$$

and finally, from Eqs. (3) and (17), α is given by

$$\alpha^2=4\sum_{j=1}^{N}(-1)^{j-1}B_j \tag{19}$$

To satisfy Eq. (16) exactly for all x in the interval $+1$ to -1, the $B_1 - B_N$ would have to be chosen so as to make the coefficients of each of the polynomials $T_2(x)$ to $T_{4N-2}(x)$ zero. Also, from Eq. (6), the $B_1 - B_N$ must satisfy

$$\sum_{j=1}^{N} (2j-1)B_j = 0 \tag{20}$$

Hence, there are $2N$ equations to be satisfied in all, and for finite N all of these cannot be satisfied exactly by a nontrivial choice of the $B_1 - B_N$. If the conditions that the coefficients of $T_{2N}(x)$ to $T_{4N-2}(x)$ should be zero are abandoned, then the $B_1 - B_N$ may be computed so as to satisfy Eq. (20), together with the $N-1$ equations obtained by comparing the coefficients of $T_2(x)$ to $T_{2N-2}(x)$. These are

$$\sum_{k=1}^{j} (2k-1)B_k = \tfrac{1}{2} \sum_{k=1}^{j} B_k B_{j-k+1} - \sum_{k=1}^{N-j} B_k B_{j+k}$$

$$j = 1, =, \ldots, N-1 \tag{21}$$

In view of the mini-max properties of the Chebsyshev polynomials [see Synder[6]] this choice of the $B_1 - B_N$ will satisfy Eq. (16) with an error whose magnitude is the minimum possible on the interval $-1 \leqslant x \leqslant +1$.

Equations (20) and (21) have been solved numerically for values of N from 2 to 20. When the magnitude of the maximum error in Eq. (16) was computed, it was found to be less than 1% for $N > 9$. The equation of the bubble surface for the case $N = 9$ was found to be

$$\begin{aligned} y = {} & 0.628t(1-x^2)^{1/2}(0.8242U_0 - 0.4661U_2 \\ & + 0.1817U_4 - 0.0728U_6 + 0.0290U_8 - 0.0116U_{10} \\ & + 0.0046U_{12} - 0.0018U_{14} + 0.0005U_{16}) \end{aligned} \tag{22}$$

and the corresponding results obtained for the vorticity and the constant h were

$$\omega = 5.048t^{-1/2} + 0(t^{1/2}) \tag{23}$$

and

$$h = 1 - 1.256t + 0(t^2) \tag{24}$$

Finally, the sum in the remainder term Eq. (11) was found to be 11.60 for $N = 9$, confirming that this term represents an error of $0(t^2)$.

It should be mentioned that Eqs. (22-24) do not represent the only solution of Eq. (21) for $N = 9$. However, all other solutions give a much larger value of the maximum error in Eq. (16) and also do not give a bubble shape which is of the smooth form depicted in Fig. 1.

For computing the bubble shape it is convenient to express Eq. (22) as a polynomial in x^2. The result is

$$\begin{aligned} y = {} & t(1-x^2)^{3/2}(1 - 4.22x^2 + 16.27x^4 - 47.67x^6 \\ & + 93.61x^8 - 112.34x^{10} + 73.53x^{12} - 19.98x^{14}) \end{aligned} \tag{25}$$

The stream function for the flow in the wake may be obtained from Eq. (10) and that for the exterior flow is found by substituting Eq. (1) into Eq. (4) and taking the imaginary part of w. The streamlines have been computed for the case $t = 0.2$ and the resulting flow pattern is shown in Fig. 2.

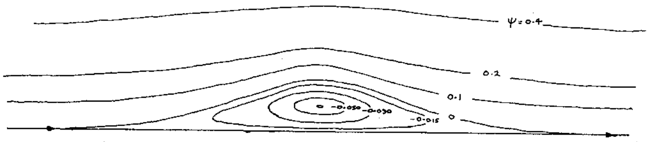

Fig. 2 Streamlines for $t = 0.2$.

Discussion

In the analysis of the preceding sections the shape of a two-dimensional slender bubble has been found [Eq. (25)] which is such that the flow exterior to the bubble is irrotational and the flow in the interior is inviscid with constant vorticity. The bubble constitutes a possible wake bubble for a shell having the same shape as the front portion of the bubble. The extent of the shell (AB Fig. 1) as a fraction of the remaining length of the bubble surface (i.e., the ratio of the length of boundary over which the standing eddy in the wake is retarded to that over which it is driven) should be such as to produce a rate of rotation which is measured by the same value of the vorticity as that given by Eq. (23). To determine this ratio would require a knowledge of the flow in the boundary layers contiguous to the bubble surface. This would involve an extensive analysis and the calculation is, therefore, not attempted here.

Conclusions

A theory is presented for the steady inviscid two-dimensional flow past a sharp nosed body. The body considered is a shell forming the front part of a slender wake bubble, the shape of which is characterized by a cusp at either end. A shape of bubble is computed which is such that the irrotational flow exterior to the bubble and the constant vorticity flow inside the bubble give continuity in pressure across the bubble boundary. The velocity increment across the bubble boundary and the strength of the vorticity are important parameters of the flow and are found to be $1 - 1.256t + 0(t^2)$ and $5.048t^{-1/2} + 0(t^{1/2})$, respectively, where t is the thickness ratio for the bubble.

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Drag Coefficient Equations for Small Particles in High Speed Flows

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Nomenclature

D = diameter, microns
C_D = drag coefficient

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